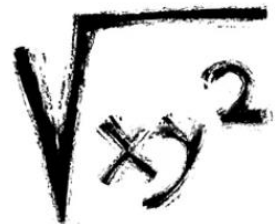

MORE OPERATIONS ON RADICALS

The chapter on fractional exponents showed us that x to the power $1/n$ is equal to the n th root of x :

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$



This way of expressing radicals using exponents allows us to prove a pair of rules we previously took for granted.

□ PROVING THE THEOREMS ABOUT RADICALS

THEOREM: Assume that x and y represent non-negative numbers. Then

$$\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$$

The root of a product
is the product of the roots.

Proof: Our technique will be to convert the left side of the formula to exponent form, use a law of exponents, and then convert back to radical form to arrive at the right side of the formula.

$$\begin{aligned} & \sqrt[n]{xy} \\ = & (xy)^{1/n} && \text{(definition of fractional exponent)} \end{aligned}$$

$$\begin{aligned}
 &= x^{1/n} y^{1/n} && \text{(law of exponents)} \\
 &= \sqrt[n]{x} \sqrt[n]{y} && \text{(definition of fractional exponent)}
 \end{aligned}$$

THEOREM: If x represents a non-negative number (that is, if $x \geq 0$), then

$$\sqrt[n]{x^n} = x$$

The n th root cancels
the n th power.

Proof: Our method of proof is similar to the previous theorem.

$$\sqrt[n]{x^n} = (x^n)^{1/n} = x^{n \cdot \frac{1}{n}} = x^1 = x$$

❑ **ADDING AND SUBTRACTING RADICALS**

We have a simple radical multiplication rule which states that $\sqrt{a}\sqrt{b} = \sqrt{ab}$. How about a rule like $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$? Seems reasonable; let's perform an experiment. Let $a = 16$ and $b = 9$.

$$\text{Left side: } \sqrt{a} + \sqrt{b} = \sqrt{16} + \sqrt{9} = 4 + 3 = 7$$

$$\text{Right side: } \sqrt{a+b} = \sqrt{16+9} = \sqrt{25} = 5 \quad \text{☹}$$

We got two different results; we thus conclude that $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$. [Of course, if $a = 0$ or $b = 0$, then it does work; but if it doesn't always work, it's not a rule. It's considered a false statement.]

So maybe we can't add $\sqrt{7}$ and $\sqrt{11}$ together, but we can add $2\sqrt{7}$ and $3\sqrt{7}$ together. This is simply the *combining of like terms*. After all, we may not be very comfortable with a number like $\sqrt{7}$, but certainly 2 of them plus 3 more of them should total 5 of them: $2\sqrt{7} + 3\sqrt{7} = 5\sqrt{7}$.

Alternatively, we can apply the distributive property to the sum $2\sqrt{7} + 3\sqrt{7}$ by factoring out the GCF: $\sqrt{7}(2 + 3)$, again giving $5\sqrt{7}$.

Thus, to add (or subtract) radicals, we can use either the concept of like terms or the idea of factoring.

EXAMPLE 1: Simplify each radical expression:

A. $3\sqrt{11} + 5\sqrt{11} = 8\sqrt{11}$

They're like terms, so we can add them up, just like
 $3x + 5x = 8x$.

Or, factor out the GCF: $\sqrt{11}(3 + 5) = 8\sqrt{11}$

B. $18\sqrt[3]{10} - 20\sqrt[3]{10} = -2\sqrt[3]{10}$

C. $\sqrt{3} + \sqrt{5}$ cannot be worked out. They're unlike terms, and no amount of simplification will turn them into like terms.

D. $3\sqrt{20} + 10\sqrt{45}$ It might appear that these terms cannot be combined, but don't jump to conclusions — a little simplification first will actually yield an answer:

$$\begin{aligned} & 3\sqrt{20} + 10\sqrt{45} \\ = & 3\sqrt{4 \cdot 5} + 10\sqrt{9 \cdot 5} \\ = & 3 \cdot 2\sqrt{5} + 10 \cdot 3\sqrt{5} \\ = & 6\sqrt{5} + 30\sqrt{5} && \text{[Like Terms!]} \\ = & 36\sqrt{5} \end{aligned}$$

E. $\sqrt{7} + \sqrt[3]{7}$ can't be simplified. Even though the radicands (the 7's) are the same, the roots are different, and therefore they are unlike terms and cannot be added together.

Homework

1. Simplify each radical expression:

a. $2\sqrt{x} + 10\sqrt{x}$

b. $3\sqrt{7} - 2\sqrt{2}$

c. $10\sqrt[3]{n} + 3\sqrt[3]{n}$

d. $6\sqrt{x} - 2\sqrt[3]{x}$

e. $9\sqrt{8} + 10\sqrt{50}$

f. $\sqrt[3]{24} + \sqrt[3]{81}$

g. $3\sqrt{80} - 5\sqrt{45}$

h. $10\sqrt[3]{189} + \sqrt[3]{56}$

i. $\sqrt[3]{z^2} + 2\sqrt[3]{z^2}$

j. $\sqrt{12} + 2\sqrt{75}$

k. $2\sqrt{20} + 2\sqrt{45}$

l. $2\sqrt{275} - 3\sqrt{99}$

❑ **MULTIPLYING RADICALS**

Using the formula $\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{xy}$, we can multiply “like roots.”

EXAMPLE 2: Simplify each radical expression:

A. $\sqrt[5]{3} \cdot \sqrt[5]{7} = \sqrt[5]{21}$

B. $\sqrt{7}\sqrt{7} = \sqrt{49} = 7$

C. $\sqrt[3]{17} \times \sqrt[3]{3} = \sqrt[3]{51}$

D. $\sqrt{2}\sqrt{8} = \sqrt{16} = 4$

Alternate Method: $\sqrt{2}\sqrt{8} = \sqrt{2}(2\sqrt{2}) = 2(\sqrt{2})^2 = 2(2) = 4$

E. $\sqrt{14}\sqrt{2} = \sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$

EXAMPLE 3: Simplify each radical expression:

A. $\sqrt{2}(1 + \sqrt{2}) = \sqrt{2} + \sqrt{2}\sqrt{2} = \sqrt{2} + 2$

$$B. \quad \sqrt{6}(\sqrt{10} - \sqrt{6}) = \sqrt{6}\sqrt{10} - \sqrt{6}\sqrt{6} = \sqrt{60} - 6 = 2\sqrt{15} - 6$$

$$C. \quad (\sqrt{3} + 1)(\sqrt{3} - 5) = \sqrt{3}\sqrt{3} - 5\sqrt{3} + 1\sqrt{3} - 5 \\ = 3 - 4\sqrt{3} - 5 = -2 - 4\sqrt{3}$$

$$D. \quad (1 + \sqrt{7})^2 = (1 + \sqrt{7})(1 + \sqrt{7}) = 1 + \sqrt{7} + \sqrt{7} + 7 = 8 + 2\sqrt{7}$$

$$E. \quad (3 + \sqrt{10})(3 - \sqrt{10}) = 9 - 3\sqrt{10} + 3\sqrt{10} - \sqrt{10}\sqrt{10} \\ = 9 - 10 = -1$$

Homework

2. Find the product:

a. $\sqrt{14}\sqrt{6}$

b. $\sqrt[3]{3} \times \sqrt[3]{9}$

c. $\sqrt{2}\sqrt{7}$

d. $\sqrt{171}\sqrt{171}$

e. $\sqrt{14}\sqrt{2}$

f. $\sqrt[3]{10}\sqrt[3]{12.5}$

g. $\sqrt[4]{2}\sqrt[4]{8}$

h. $\sqrt{x}\sqrt{yz}$

i. $\sqrt[3]{2}\sqrt[3]{4}$

3. Find the product:

a. $\sqrt{n}(1 + \sqrt{n})$

b. $(x + \sqrt{y})(x - \sqrt{y})$

c. $(2 + \sqrt{3})^2$

d. $(\sqrt{p} - q)^2$

e. $(\sqrt{7} - 2)(\sqrt{7} + 2)$

f. $(x - \sqrt{n})^2$

g. $(\sqrt{a} + \sqrt{c})(\sqrt{a} - \sqrt{c})$

h. $2(\sqrt{3} + \sqrt{4})$

i. $(\sqrt{x} + \sqrt{x})^2$

□ DIVIDING RADICALS

Just as the square root of a product is the product of the square roots

($\sqrt{ab} = \sqrt{a}\sqrt{b}$), the square root of a quotient is the quotient of the square roots.

THEOREM: Assume a and b are positive real numbers. Then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$\begin{aligned} \text{Proof: } & \sqrt{\frac{a}{b}} \\ &= \left(\frac{a}{b}\right)^{1/2} && \text{(convert radical to fractional exponent)} \\ &= \frac{a^{1/2}}{b^{1/2}} && \text{(a law of exponents)} \\ &= \frac{\sqrt{a}}{\sqrt{b}} && \text{(convert back to radical form)} \end{aligned}$$

Before we get to the heart of dividing radicals, let's recall the little formula $(\sqrt[n]{x})^n = x$. Check out the following examples:

$$\begin{aligned} (\sqrt{u})^2 &= u & (\sqrt[3]{w})^3 &= w & (\sqrt[4]{xy})^4 &= xy \\ \sqrt{c}\sqrt{c} &= (\sqrt{c})^2 = c & \sqrt{17} \times \sqrt{17} &= (\sqrt{17})^2 = 17 \end{aligned}$$

Now consider the number $\frac{1}{\sqrt{2}}$. It's just a fraction, and a calculator gives an *approximate* value of 0.707106781. But now consider the scenario where you have no calculator, but you need a decimal version of the fraction. Even if you could look up the positive square root of 2 somehow — and find that it's about 1.414213562 — you'd still be stuck with the following long division problem:

$$1.414213562 \overline{) 1.000000000}$$

This is a killer long division problem — it would take a really long time to get just a few digits of the answer.

But now watch this trick: I'm going to take the fraction $\frac{1}{\sqrt{2}}$ and

multiply it by the fraction $\frac{\sqrt{2}}{\sqrt{2}}$ (which, of course, equals 1). Here we go:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \underbrace{\left[\frac{\sqrt{2}}{\sqrt{2}} \right]}_{=1} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}$$

We've shown that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. [Double-check with your calculator.]

What's the purpose of all this? Consider the long division problem created by the second fraction, $\frac{\sqrt{2}}{2}$:

$$2 \overline{)1.414213562}$$

This is much simpler to carry out than the previous long division problem, and yet it yields the same answer. This is the historical reason why students, especially in the sciences, would convert $\frac{1}{\sqrt{2}}$ into $\frac{\sqrt{2}}{2}$. It made calculations much, much simpler.

Are you going to perform this kind of maneuver — removing radicals from the denominator of a fraction — in your Algebra course? Yes, and for two reasons. One is historical; it's been done like this for centuries. As Tevya from *Fiddler on the Roof* might say, "We do it because it's our tradition!" A second reason: It's a lot easier when we look up answers in the "back of the book" if we can all agree on what form to leave our answers in.

I'll let you in on a little secret: If you take Calculus, there will be problems where you will want to leave radicals in the denominator. Indeed, you will actually remove radicals from the numerator.

We thus have a basic agreement in math: Do not leave radicals in the denominator of a fraction. Converting a fraction with a radical in the denominator into an equivalent

Some trig books will leave answers like $\frac{1}{\sqrt{3}}$, so be prepared.

fraction without a radical in the denominator is called ***rationalizing the denominator***.

EXAMPLE 4:

- A. Rationalize the denominator, which is a **monomial**:

$$\frac{3}{\sqrt{x}} = \frac{3}{\sqrt{x}} \left[\frac{\sqrt{x}}{\sqrt{x}} \right] = \frac{3\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{3\sqrt{x}}{x}$$

$= 1$

- B. Rationalize the denominator, which is a **binomial**:

$$\frac{\sqrt{2}}{5 - \sqrt{10}}$$

For this example, the trick is in figuring out what, exactly, we should multiply the top and the bottom of the fraction by.

$$\frac{\sqrt{2}}{5 - \sqrt{10}} = \frac{\sqrt{2}}{5 - \sqrt{10}} \left[\frac{5 + \sqrt{10}}{5 + \sqrt{10}} \right] = \frac{5\sqrt{2} + \sqrt{20}}{25 - 10} = \frac{5\sqrt{2} + 2\sqrt{5}}{15}$$

$= 1$

Notes: Many students try multiplying top and bottom by $\sqrt{10}$, figuring that it would remove the $\sqrt{10}$. Try it and you'll see this procedure fail.

Other students think they're more clever and multiply top and bottom by $5 - \sqrt{10}$. Go for it; you'll still have radicals in the bottom.

The quantity $5 + \sqrt{10}$, which does eradicate the radical from the bottom, is called the ***conjugate*** of $5 - \sqrt{10}$.

Homework

4. Simplify so that there are no radicals in the denominator:

a. $\frac{2}{\sqrt{8}}$ b. $\frac{x^2}{\sqrt{x}}$ c. $\frac{3}{5+\sqrt{2}}$ d. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

e. $\frac{1}{\sqrt{3}}$ f. $\frac{6}{\sqrt{9}}$ g. $\frac{8}{\sqrt{2}}$ h. $\frac{\sqrt{x}}{\sqrt{y}}$

i. $\frac{5}{1+\sqrt{5}}$ j. $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ k. $\frac{7}{3\sqrt{2}}$ l. $\frac{x}{n\sqrt{x}}$

m. $\frac{7}{1+\sqrt{7}}$ n. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$ o. $\frac{a}{b\sqrt{c}}$ p. $\frac{7+\sqrt{103}}{7+\sqrt{103}}$

5. Two of the following are valid, while two of them are bogus. Which are which?

a. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

b. $\sqrt{a-b} = \sqrt{a} - \sqrt{b}$

c. $\sqrt{ab} = \sqrt{a}\sqrt{b}$

d. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Review Problems

6. a. Find the two fourth roots of 256.
 b. Find all the cube roots of -1000.
 c. Find all the square roots of -121.

7. Simplify each expression:

a. $\sqrt{7}\sqrt{5}$ b. $\sqrt{7}+\sqrt{5}$ c. $\frac{\sqrt{7}}{\sqrt{5}}$ d. $\sqrt{7}-\sqrt{5}$

e. $\sqrt{2} + \sqrt[3]{2}$ f. $\sqrt{28} + \sqrt{175}$ g. $(\sqrt{h} + 7)^2$ h. $(3 - \sqrt{t})^2$

8. Simplify so that there are no radicals in the denominator:

a. $\frac{4}{\sqrt{35} - 5}$ b. $\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$ c. $\frac{12}{\sqrt{9} + 3}$

9. Simplify each expression:

a. $\sqrt{18}\sqrt{8}$ b. $\sqrt{7} - \sqrt{7}$ c. $\frac{1 + \sqrt{7}}{\sqrt{7}}$ d. $\sqrt{50} - \sqrt{8}$
 e. $\frac{3}{\sqrt{28} + 5}$ f. $\frac{\sqrt{a} + b}{\sqrt{a} - b}$ g. $\frac{3 - \sqrt{25}}{\sqrt{25} - 3}$

10. Let $r_1 = \frac{3 + \sqrt{11}}{4}$ and $r_2 = \frac{3 - \sqrt{11}}{4}$.

Prove a. $r_1 + r_2 = \frac{3}{2}$ b. $r_1 \times r_2 = -\frac{1}{8}$

11. True/False:

- 36 has two square roots.
- $\sqrt{49} = \pm 7$
- If $x \leq 0$, then $\sqrt{x}$ is not a real number.
- 64 has exactly one cube root.
- $-\sqrt[4]{20}$ is a real number.
- $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ Assume $a, b \geq 0$
- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ Assume $a, b \geq 0$
- $\sqrt[3]{18} = 3\sqrt{2}$
- $\sqrt{20} + \sqrt{5} = 5$
- $\sqrt{20} + \sqrt{5} = 3\sqrt{5}$
- $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$
- $\frac{7}{\sqrt{n}} = 7\sqrt{n}$

“The wisest mind has something yet to learn.”

George Santayana (1863 - 1952)

m. $\frac{1}{\sqrt{n}+1} = \frac{\sqrt{n}-1}{n-1}$

n. $\sqrt[3]{x^2} + \sqrt[3]{x} = x$

□ TO ∞ AND BEYOND

A. Show that $\frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

Note: A calculator gives good evidence, but it's not a proof.

B. Show that $2\frac{r}{\sqrt{2}}\sqrt{r^2-\frac{r^2}{2}} = r^2$

Solutions

1. a. $12\sqrt{x}$ b. As is c. $13\sqrt[3]{n}$ d. As is e. $68\sqrt{2}$
 f. $5\sqrt[3]{3}$ g. $-3\sqrt{5}$ h. $32\sqrt[3]{7}$ i. $3\sqrt[3]{z^2}$ j. $12\sqrt{3}$
 k. $10\sqrt{5}$ l. $\sqrt{11}$

2. a. $\sqrt{84} = 2\sqrt{21}$ b. $\sqrt[3]{27} = 3$ c. $\sqrt{14}$
 d. 171 e. $\sqrt{28} = 2\sqrt{7}$ f. $\sqrt[3]{125} = 5$
 g. $\sqrt[4]{16} = 2$ h. $\sqrt{xyz}$ i. $\sqrt[3]{8} = 2$

3. a. $\sqrt{n}+n$ b. $x^2-x\sqrt{y}+x\sqrt{y}-y = x^2-y$
 c. $4+4\sqrt{3}+3 = 7+4\sqrt{3}$ d. $(\sqrt{p}-q)(\sqrt{p}-q) = p-2q\sqrt{p}+q^2$
 e. $7-4 = 3$ f. $x^2-2x\sqrt{n}+n$ g. $a-c$
 h. $2(\sqrt{3}+2) = 2\sqrt{3}+4$ i. $(2\sqrt{x})^2 = 4x$

4. a. $\frac{2}{\sqrt{8}} \left[\frac{\sqrt{8}}{\sqrt{8}} \right] = \frac{2\sqrt{8}}{8} = \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2}$ b. $x\sqrt{x}$

c. $\frac{3}{5+\sqrt{2}} \left[\frac{5-\sqrt{2}}{5-\sqrt{2}} \right] = \frac{15-3\sqrt{2}}{25-2} = \frac{15-3\sqrt{2}}{23}$ d. $\frac{x+2\sqrt{xy}+y}{x-y}$

e. $\frac{\sqrt{3}}{3}$ f. 2 g. $4\sqrt{2}$ h. $\frac{\sqrt{xy}}{y}$

i. $\frac{5}{1+\sqrt{5}} \left[\frac{1-\sqrt{5}}{1-\sqrt{5}} \right] = \frac{5-5\sqrt{5}}{1-5} = \frac{5-5\sqrt{5}}{-4} = \frac{5-5\sqrt{5}}{-4} \left[\frac{-1}{-1} \right] = \frac{5\sqrt{5}-5}{4}$

The last step is optional. It was done just to remove the negative sign from the bottom of the fraction.

j. $-3-2\sqrt{2}$ k. $\frac{7}{3\sqrt{2}} \left[\frac{\sqrt{2}}{\sqrt{2}} \right] = \frac{7\sqrt{2}}{3 \cdot 2} = \frac{7\sqrt{2}}{6}$ l. $\frac{\sqrt{x}}{n}$

m. $\frac{7\sqrt{7}-7}{6}$ n. $\frac{a^2+2a\sqrt{b}+b}{a^2-b}$ o. $\frac{a\sqrt{c}}{bc}$ p. 1

5. c. and d. are valid — a. and b. are bogus.

6. a. ± 4 b. -10 c. None

7. a. $\sqrt{35}$ b. As is c. $\frac{\sqrt{35}}{5}$ d. As is

e. As is f. $7\sqrt{7}$ g. $h+14\sqrt{h}+49$ h. $9-6\sqrt{t}+t$

8. a. $\frac{2\sqrt{35}+10}{5}$ b. $\frac{a-2\sqrt{ab}+b}{a-b}$ c. 2

9. a. 12 b. 0 c. $\frac{\sqrt{7}+7}{7}$ d. $3\sqrt{2}$

e. $2\sqrt{7}-5$ f. $\frac{a+2b\sqrt{a}+b^2}{a-b^2}$ g. -1

10. Do the arithmetic and it should work out.

11. a. T b. F c. F d. T e. T f. F g. T h. F
i. F j. T k. T l. F m. T n. F